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Relativistic dynamics of charges in external fields: the Pauli algebra approach

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Abstract. The Pauli algebra is used to study the relativistic motion of charged particles in external electromagnetic fields. Plane-wave solutions to Maxwell's equations, in particular standing waves with electric and magnetic fields which are everywhere parallel, are easily represented and analysed in a gauge-independent treatment. We derive the 4-momentum of electromagnetic fields and show how the integrals for fields of a charged particle can be evaluated with integration limits specified in the rest frame of the moving charge. The general solution of the Lorentz force equation is shown to be a Lorentz transformation with time-dependent rotation and boost parameters which, in the case of parallel fields, are trivially related to the magnetic and electric fields, respectively. With the Pauli algebra, an earlier derivation in the Dirac algebra of the 4-velocity in constant uniform fields is greatly simplified and extended to arbitrary initial motion, and solutions are found to motion in a Coulomb field and in the presence of arbitrarily polarised (monochromatic, directed) plane waves.

1. Introduction

In a previous paper (Baylis and Jones 1989), we have seen how the Pauli algebra $\mathscr{P}$, as the Clifford algebra in Euclidean 3-space is known, provides a natural framework for component-free covariant treatments of problems in special relativity. Minkowski-space 4-vectors and antisymmetric second-rank tensors are simple elements of $\mathscr{P}$, which in any given inertial reference frame are easily written as the sum of time ($\mathscr{P}$-scalar) and space (vector) parts. Higher-rank tensors appear in contracted form. Although $\mathscr{P}$ can be represented by 2×2 matrices, the algebraic products of vectors and scalars are so simple that Lorentz transformations and other manipulations of elements are easily found without reference to matrices or components. The formalism we have developed is applicable to a wide range of problems in relativistic physics and allows straightforward comparisons to formulations in the Dirac algebra or to traditional work with Minkowski-space components.

We have already applied the Pauli algebra approach to relativistic kinematics and to problems involving compounded Lorentz transformations (Baylis and Jones 1988). However, some of the most impressive applications of the approach are to classical electrodynamics. The purpose of this paper is to present some of these applications.

The Dirac algebra D, the Clifford algebra for products of Minkowski-space 4-vectors, seems the more natural choice for problems in special relativity, and it has been extensively investigated for this purpose (Chisholm and Common 1986, Grieder 1984, Hestenes 1966, 1974, Salingaros 1985a, b, 1986a, Salingaros and Dresden 1983). However, the smaller $\mathscr{P}$, which is isomorphic to the even half D_+ of D, is capable of

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handling essentially all physical applications of D with considerable advantage in economy (Baylis and Jones 1988, 1989).

The power of the Dirac algebra has been recently demonstrated by a calculation of the relativistic motion of a charge starting from rest in an arbitrary but constant and uniform electromagnetic field (Salingaros 1985a). In this paper, we repeat and extend these results using the Pauli algebra to cases of arbitrary initial motion of the charge and to motion in a Coulomb field and in an arbitrarily polarised plane wave. We also extend a result of Rohrlich (1960) dealing with the flow of energy-momentum in a classical model of a charged particle.

In § 2 the basic equations of electrodynamics are given and the energy-momentum density of fields is derived. A Lorentz transformation of a differential operator is illustrated here. We also analyse plane waves, including standing waves with parallel electric and magnetic fields, in a simple gauge-independent treatment. In § 3, solutions of the Lorentz force equations are found and are presented as time-dependent Lorentz transformations of the initial 4-velocity. For a magnetic field, the transformation is a pure rotation, whereas for an electric field of fixed direction, the transformation is a pure boost. Results are summarised in § 4.

2. Basic equations of electrodynamics

2.1. Maxwell's equations

In $\mathscr{P}$, Maxwell's equations are embodied in the single equation

$$\bar{\partial} F = 4\pi \bar{J} \tag{1}$$

which relates the electromagnetic field

$$F = \partial \bar{A} = \boldsymbol{E} + \mathrm{i}\boldsymbol{B} \tag{2}$$

to the 4-current density $J = \rho + \boldsymbol{J}$ through the differential operator $\bar{\partial} = \partial_t + \boldsymbol{\nabla}$ (Baylis and Jones 1989). Here the vector potential $A = \phi + \boldsymbol{A}$ is assumed to obey the Lorentz condition

$$\bar{\partial} \cdot A = \partial \cdot \bar{A} = \partial_t \phi + \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0 \tag{3}$$

and we have used Gaussian units with $c = 1$. The covariance of (1) follows from the behaviour of 4-vectors (like ∂, A and J) and 6-vectors (like F) under restricted Lorentz transformations $L = \exp(\boldsymbol{w}/2 - \mathrm{i}\boldsymbol{\theta}/2)$:

$$A \to LAL^{+} \qquad F \to LF\bar{L}. \tag{4}$$

Any element in $\mathscr{P}$ can be uniquely decomposed into four parts: (real) scalar, pseudoscalar, (real) vector and pseudovector. The four corresponding parts of (1) give the usual Maxwell's equations for $\boldsymbol{\nabla} \cdot \boldsymbol{E}$, $\boldsymbol{\nabla} \cdot \boldsymbol{B}$, $\boldsymbol{\nabla} \times \boldsymbol{B}$ and $\boldsymbol{\nabla} \times \boldsymbol{E}$, respectively, as is easily seen when $\bar{\partial} F = (\partial_t + \boldsymbol{\nabla})(\boldsymbol{E} + \mathrm{i}\boldsymbol{B})$ is expanded according to the simple rule for vector products in $\mathscr{P}$:

$$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \mathrm{i}\boldsymbol{a} \times \boldsymbol{b}. \tag{5}$$

Because multiplication in $\mathscr{P}$ is associative, Maxwell's equations (1) can be expressed equivalently by the wave equation

$$\Box A \equiv \partial \bar{\partial} A = 4\pi J. \tag{1a}$$

Here J is the electric current density. A current of magnetic monopoles could be added by replacing J by the complex element $J \to J^{(e)} + iJ^{(m)}$, whose real and imaginary parts give the electric and magnetic components, respectively. According to ($1a$), the 4-potential A then also becomes complex. The rule (Baylis and Jones 1989) that physical 4-vectors be represented by real elements of $\mathcal{P}$ seems to exclude magnetic monopoles. However, even without this rule, the absence of magnetic charges is indistinguishable from the universal existence of a magnetic charge in a constant ratio, say $\tan\phi$, to the electric charge, since then J can be replaced by $J\exp(i\phi)$, where J is real, and the phase factor $\exp(i\phi)$ can be removed from Maxwell's equation (1) by a duality rotation (Baylis and Jones 1989). The continuity equation $\partial\cdot\bar{J}=0$ results simply from Maxwell's equation (1) and the vanishing of the scalar part of $\Box F$.

2.2. *Plane-wave solutions*

Solutions to Maxwell's equation (1) in source-free ($J=0$) space may be written as linear combinations of plane-wave solutions, each with a definite propagation 4-vector k. In these plane-wave solutions, the spacetime dependence of the field is given by the scalar $\bar{k}\cdot x = \omega t - \boldsymbol{k}\cdot\boldsymbol{x}$:

$$F = F(\bar{k}\cdot x). \tag{6}$$

The wave equation for F, namely $\Box F = 0$, then gives

$$k\bar{k} \equiv \omega^2 - \boldsymbol{k}^2 = 0 \tag{7}$$

and Maxwell's equation (1) becomes an eigenvalue equation

$$\hat{k}F = F. \tag{8}$$

Since the scalar part of F is zero, $\hat{k}\cdot F = 0$ and F must have the form

$$\begin{aligned} F = \boldsymbol{F} &= \tfrac{1}{2}(1+\hat{k})\hat{\xi}f(\bar{k}\cdot x) \\ &= \tfrac{1}{2}(\hat{\xi}+i\hat{\eta})f(\bar{k}\cdot x) \end{aligned} \tag{9}$$

where $\hat{\xi}$ is any real unit vector in the plane normal to $\hat{k}$, $\hat{\eta} = \hat{k}\times\hat{\xi}$ and f is an arbitrary scalar function. We note in passing that $\frac{1}{2}(1+\hat{k})$ is the eigenprojector of $\bar{k}$ with eigenvalue 0 (Baylis and Jones 1989).

A rotation is a Lorentz transformation (4) for which L is unitary. A rotation of the plane wave (9) about the direction of propagation $\hat{k}$ is especially simple:

$$F \to \exp(-i\phi\hat{k}/2)F\exp(i\phi\hat{k}/2) = \exp(-i\phi\hat{k})F = \exp(-i\phi)F. \tag{10}$$

Here we first used the fact that $\hat{k}$ and $\boldsymbol{F}$ are perpendicular and hence anticommute (see (5)) and then we applied the eigenvalue equation (8). Consequently, if $f(\bar{k}\cdot x)$ is proportional to $\exp(\pm i\bar{k}\cdot x)$, the solution (9) describes a circularly polarised plane wave of helicity ± 1, i.e. one in which at any fixed position $\boldsymbol{x}$, the field F rotates about $\hat{k}$ at the angular frequency $\pm\omega$. On the other hand, if the complex phase of f is constant, the solution (9) represents a linearly polarised plane wave.

More generally, the solution (9) represents a directed monochromatic plane wave whose polarisation may be circular, linear or elliptical. Its 4-momentum density is the constant 4-vector

$$FF^{+}/8\pi = \tfrac{1}{4}(1+\hat{k})\hat{\xi}\hat{\xi}(1+\hat{k})|f|^2/8\pi = \tfrac{1}{2}(1+\hat{k})|f|^2/8\pi \tag{11}$$

with a non-vanishing vector part and its Lorentz invariants $\boldsymbol{E}^2-\boldsymbol{B}^2$ and $\boldsymbol{E}\cdot\boldsymbol{B}$ are identially zero:

$$F^2=\boldsymbol{E}^2-\boldsymbol{B}^2+2\mathrm{i}\boldsymbol{E}\cdot\boldsymbol{B}=\tfrac{1}{4}(1+\hat{k})\hat{\xi}(1+\hat{k})\hat{\xi}f^2=0. \tag{12}$$

Note that the integral $\frac{1}{2}\int \mathrm{d}^4x\, F^2=S-\mathrm{i}Q$ gives the action S and pseudocharge Q (Brownstein 1986). The functional form of $f(\bar{k}\cdot x)$ and hence its polarisation is invariant under restricted Lorentz transformations (4).

Standing plane waves can be formed by superimposing one plane wave with propogation vector $k=\omega+\boldsymbol{k}$ with another with propogation vector $\bar{k}=\omega-\boldsymbol{k}$. Therefore, they have the form

$$F=\tfrac{1}{2}(1+\hat{k})\hat{\xi}f_1(\bar{k}\cdot x)+\tfrac{1}{2}(1-\hat{k})\hat{\xi}f_2(k\cdot x). \tag{13}$$

Such a wave is monochromatic only in frames with no longitudinal motion ($\boldsymbol{v}\cdot\hat{k}=0$) with respect to the laboratory frame. The 4-momentum density of such a wave is

$$\frac{FF^+}{8\pi}=\frac{1}{16\pi}[(1+\hat{k})|f_1|^2+(1-\hat{k})|f_2|^2] \tag{14}$$

and the Lorentz invariants (compare (12)) are

$$F^2=\tfrac{1}{2}(1+\hat{k})\hat{\xi}(1-\hat{k})\hat{\xi}f_1f_2=f_1(\bar{k}\cdot x)f_2(k\cdot x). \tag{15}$$

2.3. *Waves with parallel electric and magnetic fields*

Chu and Ohkawa (1982) have recently pointed out the existence of standing waves whose electric and magnetic fields are everywhere parallel. In spite of arguments that such waves do not exist (Lee 1983, Salingaros 1985b, 1986b), it now seems clear that they do (Chu 1983, Zaghloul *et al* 1987, 1988, Chu and Ohkawa 1987). Their existence and properties are easily examined with the aid of our Pauli algebra approach. In contrast to earlier treatments, ours is gauge invariant.

The property that $\boldsymbol{E}$ is everywhere parallel to $\boldsymbol{B}$ is established by the condition that $\boldsymbol{E}\times\boldsymbol{B}$, and hence the vector part of the 4-momentum density, vanishes identically. From (14), this condition is fulfilled if and only if $|f_1(\bar{k}\cdot x)|=|f_2(k\cdot x)|$ at all x. Since the scalar arguments $\bar{k}\cdot x$ and $k\cdot x$ are different functions of x, it is necessary and sufficient that the two superimposed plane waves be pure circularly polarised waves of the same amplitude:

$$f_j(s)=f_0\exp(-\mathrm{i}\lambda_j s) \qquad j=1,2 \tag{16}$$

where λ_j is the helicity of the jth directed plane wave. (We have chosen the arbitrary origin $x=0$ to be at a position where the phases of the oppositely propagating plane waves are equal.) The 4-momentum density for such waves (14) has only an energy part

$$FF^+/8\pi=|f_0|^2/8\pi \tag{17}$$

and the nature of the Lorentz invariants F^2 depends on whether $\lambda_1=\lambda_2$ or $\lambda_1=-\lambda_2$. There are thus two types of standing plane wave with parallel electric and magnetic fields and there are two 'helicities' associated with each type.

Type-I waves have their component helicities equal: $\lambda_1=\lambda_2\equiv\lambda=\pm1$. The direction of the electromagnetic field F of such waves is frozen in time but rotates as a function of $\boldsymbol{k}\cdot\boldsymbol{x}$:

$$\begin{aligned}F_{\mathrm{I}}&=f_0\exp(-\mathrm{i}\lambda\omega t)(\hat{\xi}\cos\boldsymbol{k}\cdot\boldsymbol{x}-\lambda\hat{\eta}\sin\boldsymbol{k}\cdot\boldsymbol{x})\\&=f_0\exp[-\mathrm{i}\lambda(\omega t-\hat{k}\boldsymbol{k}\cdot\boldsymbol{x})]\hat{\xi}\end{aligned} \tag{18}$$

whereas its Lorentz-invariant square

$$F_{\mathrm{I}}^2 = f_0^2 \exp(2\mathrm{i}\lambda\omega t) \tag{19}$$

is uniform in space but oscillates in time. The individual electric- and magnetic-field amplitudes oscillate in time and are a quarter-cycle out of phase.

Type-II waves have components of opposite helicities: $\lambda_2 = -\lambda_1 \equiv \lambda = \pm 1$. The electromagnetic field now has a direction which is the same at all positions but rotates in time:

$$\begin{aligned} F_{\mathrm{II}} &= f_0 \exp(\mathrm{i}\lambda \boldsymbol{k}\cdot\boldsymbol{x})(\hat{\xi}\cos\omega t + \hat{\eta}\sin\omega t) \\ &= f_0 \exp[\mathrm{i}\lambda(\boldsymbol{k}\cdot\boldsymbol{x} - kt)]\hat{\xi} \end{aligned} \tag{20}$$

whereas its Lorentz-invariant square

$$F_{\mathrm{II}}^2 = f_0^2 \exp(2\mathrm{i}\lambda \boldsymbol{k}\cdot\boldsymbol{x}) \tag{21}$$

is frozen in time but oscillates as a function of $\boldsymbol{k}\cdot\boldsymbol{x}$. Note that to derive the second more compact relations for F_{I} (19) and F_{II} (20), we used a simple identity for the rotations of $\hat{\xi}$ about $\hat{k}$ by the angle ϕ:

$$\exp(-\mathrm{i}\phi\hat{k}/2)\hat{\xi}\exp(\mathrm{i}\phi\hat{k}/2) = \exp(-\mathrm{i}\phi\hat{k})\hat{\xi} = \hat{\xi}\cos\phi + \hat{\eta}\sin\phi. \tag{22}$$

The waves proposed by Chu and Ohkawa (1982) are type-I waves with $\lambda = +1$. The generalisation of Zaghloul *et al* (1987) includes also type-II waves (Chu and Ohkawa 1987), and indeed their condition relating magnitudes of derivatives of the vector-potential components is equivalent to our condition on $|f_j(s)|$. (Whether or not the derivatives have parallel directions is not meaningful for the real vector potentials since they rotate in spacetime with different x dependencies.) We also see that the average of the action and pseudocharge of type-I and II waves vanishes as expected (Brownstein 1986). However, superpositions of type-II waves do exist with vanishing DC components but finite action S and pseudocharge Q. The field

$$F = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\omega f(\omega)\exp[\mathrm{i}\lambda(\boldsymbol{k}\cdot\boldsymbol{x} - kt)]\xi \qquad f(0) = 0$$

has $S - \mathrm{i}Q = (\pi\lambda/4)[f'(0)]^2$, where $f'(0)$ is the slope of $f(\omega)$ in the limit $\omega \to 0$. That such waves are of little practical value follows from the property by which they escape the conditions of Brownstein's (1986) proof of vanishing $S - \mathrm{i}Q$: they are spatially unbounded at all times t.

2.4. *The Lorentz force equation*

The Lorentz force equation is easily derived from the rest-frame force law

$$m\frac{\mathrm{d}u_{\mathrm{rest}}}{\mathrm{d}\tau} = q\boldsymbol{E} = \tfrac{1}{2}q(\boldsymbol{F}_{\mathrm{rest}} + \boldsymbol{F}_{\mathrm{rest}}^{+}) \tag{23}$$

where $u = \gamma + \boldsymbol{u}$ is the 4-velocity and τ is the proper time. Equation (23) is *not* covariant: it is valid only in the frame which is instantaneously at rest with the particle. However, noting that $\boldsymbol{F}_{\mathrm{rest}} = \bar{L}FL$, $L\bar{L} = 1$ and $LL^{+} = u$, we obtain

$$\begin{aligned} m\frac{\mathrm{d}u}{\mathrm{d}\tau} &= Lm\frac{\mathrm{d}u_{\mathrm{rest}}}{\mathrm{d}\tau}L^{+} = \tfrac{1}{2}qL(\bar{L}FL + L^{+}F^{+}\bar{L}^{+})L^{+} \\ &= \tfrac{1}{2}q(\boldsymbol{F}u + u\boldsymbol{F}^{+}) \end{aligned} \tag{24}$$

which is the correct covariant generalisation of (23). The result is also readily obtained from a Lagrangian or Hamiltonian formalism, from which one finds the suggestive covariant form

$$\mathrm{d}\pi/\mathrm{d}\tau = \partial(\bar{\pi}\cdot u) = q\partial(\bar{A}\cdot u) \tag{25}$$

where $\pi = mu + qA$ (with A evaluated at the position of the charge) is the conjugate momentum and, on the right-hand side, the 4-velocity u and position x are treated as independent variables. The Lorentz scalar $\bar{A}\cdot u$ is just the electric potential ϕ in the rest frame of the charge. Consequently, (25) is a generalisation of the non-relativistic electrostatic force law, in which the force is $-\nabla\phi$. Equation (24) follows directly when one notes

$$\frac{\mathrm{d}\pi}{\mathrm{d}\tau} = m\frac{\mathrm{d}u}{\mathrm{d}\tau} + q\frac{\mathrm{d}A}{\mathrm{d}\tau} = m\frac{\mathrm{d}u}{\mathrm{d}\tau} + qu\cdot\partial A \tag{26}$$

and expands the Lorentz scalars $\bar{A}\cdot u$ and $u\cdot\bar{\partial} = \gamma\partial_t + \boldsymbol{u}\cdot\nabla$ according to

$$\bar{A}\cdot u = \tfrac{1}{2}(\bar{A}u + \bar{u}A) \qquad u\cdot\bar{\partial} = \tfrac{1}{2}(u\bar{\partial} + \partial\bar{u}). \tag{27}$$

2.5. Momentum conservation

Momentum conservation for a system of sources and their associated fields follows directly from Maxwell's equations and the Lorentz force equation. The total momentum p of a system of charges is found by integrating (24) and summing:

$$p = \sum_i \tfrac{1}{2}q_i \int \mathrm{d}\tau_i(\boldsymbol{F}_i u_i + u_i\boldsymbol{F}_i^+) = \tfrac{1}{2}\int \mathrm{d}^4x(\boldsymbol{F}J + J\boldsymbol{F}^+) \tag{28}$$

where d^4x is the Lorentz-invariant 4-volume element and J is the current density

$$J(x) = \sum_i \int_{-\infty}^{\infty} \mathrm{d}\tau_i\, q_i u_i \delta^{(4)}[x - r_i(\tau)]. \tag{29}$$

The integrand on the right-hand side of (28),

$$(\boldsymbol{F}J + J\boldsymbol{F}^+)/2 = \boldsymbol{J}\cdot\boldsymbol{E} + \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B} \tag{30}$$

may be interpreted as the time-rate of change of the kinetic energy-momentum density of the particles. When the covariant expression (28) for p is evaluated in any inertial frame, the time integration is indefinite and the calculated 4-momentum is that from the currents and fields inside the volume over which the spatial integration is performed. From Maxwell's equation (1),

$$\tfrac{1}{2}(\boldsymbol{F}J + J\boldsymbol{F}^+) = -[\boldsymbol{F}(\partial\boldsymbol{F}^+) + (\partial\boldsymbol{F}^+)\boldsymbol{F}^+]/8\pi = -(\boldsymbol{F}\partial\boldsymbol{F}^+)/8\pi \tag{31}$$

where the derivative operates both forward and backward within the parenthesis, and

$$\partial\boldsymbol{F}^+ = \boldsymbol{F}\partial \tag{32}$$

is a real 4-vector. Now the 4-momentum density $U + \boldsymbol{S}$ and Maxwell stress tensor $\overleftrightarrow{T}$ are given by the Hermitian quantities

$$\boldsymbol{F}\boldsymbol{F}^+ = 8\pi(U + \boldsymbol{S}) \qquad \boldsymbol{F}\hat{\boldsymbol{n}}\boldsymbol{F}^+ = 8\pi(\hat{\boldsymbol{n}}\cdot\overleftrightarrow{T} - \hat{\boldsymbol{n}}\cdot\boldsymbol{S}). \tag{33}$$

Consequently,

$$(\boldsymbol{F}\partial\boldsymbol{F}^+)/8\pi = \frac{\partial}{\partial t}(U + \boldsymbol{S}) - \nabla\cdot\overleftrightarrow{T} + \nabla\cdot\boldsymbol{S} \tag{34}$$

and (31) is simply a compact expression of the energy-momentum conservation of currents J and their associated fields (Jackson 1975).

Substitution of (31) and (34) into (28) and partial integration (Gauss's law) gives an integral expression for the total 4-momentum of the particles:

$$\begin{aligned} p &= -\frac{1}{8\pi}\int d^4x(\boldsymbol{F}\partial\boldsymbol{F}^+) \\ &= -\int d^3x(U+\boldsymbol{S})+\int dt \oint d\boldsymbol{\Sigma}\cdot(\bar{T}-\boldsymbol{S}) \end{aligned} \tag{35}$$

where $d\boldsymbol{\Sigma}$ is a surface element and the integration is to be performed in the lab frame.

For studying classical charges of finite size, however, one may want the volume and surface integrals to be referenced to the rest frame of the charge, as, for example, when the classical charge distribution vanishes inside a small sphere in the rest frame. The derivation is again straightforward, but now the differential operator in (35) needs to be expressed in terms of the rest-frame coordinates:

$$\partial = L\partial_{\text{rest}}L^+ \tag{36}$$

so that the result of partial integration is expressed in terms of rest-frame integration elements. The result, where u is the 4-velocity of the particle, is a generalisation of Rohrlich's (1960) result in that now the surface terms are included:

$$\begin{aligned} p = &-\int d^3x_{\text{rest}}[\gamma U-\boldsymbol{u}\cdot\boldsymbol{S}+\gamma\boldsymbol{S}+\boldsymbol{u}\cdot\vec{\bar{T}}] \\ &+\int dt\oint d\boldsymbol{\Sigma}\cdot[\boldsymbol{u}U-\boldsymbol{S}-(\gamma-1)\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\cdot\boldsymbol{S}+\boldsymbol{u}\boldsymbol{S}+\vec{\bar{T}}+(\gamma-1)\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\cdot\vec{\bar{T}}]. \end{aligned} \tag{37}$$

In (37) we recall the rule (Baylis and Jones 1989) that, when various multiplication types appear together in the same level of parenthesis, scalar (dot) products are to be evaluated before normal algebraic products.

3. Solutions of the Lorentz force equation

The formal solution of the Lorentz force equation (24) is a time-dependent Lorentz transformation of $u(0)$:

$$u(\tau) = L(\tau)u(0)L(\tau)^+ \tag{38}$$

where the transformation is a unimodular element satisfying

$$\frac{d}{d\tau}L(\tau) = \frac{q}{2m}\boldsymbol{F}(\tau)L(\tau) \tag{39}$$

which can be written as an exponential

$$L(\tau) = \exp\left(\frac{q}{2m}\int_0^\tau d\tau_1\, \boldsymbol{F}(\tau_1)\right). \tag{40}$$

Here $\boldsymbol{F}(\tau)$ is the electromagnetic field at the position of the charge at proper time τ. Generally it is known only once u is determined and integrated. If $\boldsymbol{F}$ changes direction as a function of proper time, then since vectors in different directions do not commute, L (equation (40)) must be treated as a time-ordered exponential:

$$L(\tau)=1+\frac{q}{2m}\int_0^\tau \mathrm{d}\tau_1\,\boldsymbol{F}(\tau_1)+\left(\frac{q}{2m}\right)^2\int_0^\tau \mathrm{d}\tau_1\int_0^{\tau_1}\mathrm{d}\tau_2\,\boldsymbol{F}(\tau_1)\boldsymbol{F}(\tau_2)+\ldots. \tag{41}$$

Of course if the (complex) direction of $\boldsymbol{F}$ is constant, then $\boldsymbol{F}$ at different times commute and the Dyson expansion (41) is identical to the simple exponential,.

Consider first several special cases in which $\boldsymbol{F}$ has a constant direction.

3.1. Parallel electric and magnetic fields

The Lorentz transformation is the product of a commuting boost and rotation (a 'rifle' transformation), where the boost parameter is a linear function of the electric field

$$\boldsymbol{w}=\frac{q}{m}\int_0^\tau \mathrm{d}\tau_1\,\boldsymbol{E}(\tau_1) \tag{42}$$

and the rotation angle is a linear function of the magnetic field

$$\boldsymbol{\theta}=-\frac{q}{m}\int_0^\tau \mathrm{d}\tau\boldsymbol{B}(\tau_1). \tag{43}$$

3.2. Arbitrary constant and uniform fields

A class of inertial frames can always be found in which the electric and magnetic fields are parallel or in which one of the fields vanishes; the solution of § 3.1 can then be used with the appropriate boost. Alternatively, the transformation (38) can be calculated directly by expanding (see (10) of Baylis and Jones (1989))

$$\exp\left(\frac{q}{2m}\boldsymbol{F}\tau\right)=\cosh\left(\frac{q}{2m}f\tau\right)+\boldsymbol{F}\sinh\left(\frac{q}{2m}f\tau\right)f^{-1} \tag{44}$$

where here the scalar f is the Lorentz-invariant complex length of $\boldsymbol{F}$

$$f^2=\boldsymbol{F}\boldsymbol{F}=E^2-B^2+2\mathrm{i}\boldsymbol{E}\cdot\boldsymbol{B} \tag{45}$$

whose real and imaginary parts $\varepsilon=\mathscr{R}f$, $\beta=\mathscr{I}f$ are just the magnitudes of the electric and magnetic fields in any frame where these are parallel. (They can be easily expressed in terms of the Lorentz invariant quantities of (45).) The resulting solution for arbitrary initial 4-velocity u_0 is

$$u=\tfrac{1}{2}u_0(\cosh\varepsilon s+\cos\beta s)+\frac{[\cdot]}{(\varepsilon^2+\beta^2)^{1/2}} \tag{46}$$

where

$$\begin{aligned}[\cdot]&=\mathscr{R}[\boldsymbol{F}u\boldsymbol{F}^+(\sinh\varepsilon s+\mathrm{i}\sin\beta s)]+\tfrac{1}{2}\boldsymbol{F}u_0\boldsymbol{F}^+(\cosh\varepsilon s-\cos\beta s)\\
&=(\gamma_0\boldsymbol{E}+\boldsymbol{u}_0\cdot\boldsymbol{E}+\boldsymbol{u}_0\times\boldsymbol{B})(\varepsilon\sinh\varepsilon s+\beta\sin\beta s)\\
&\quad+(\gamma_0\boldsymbol{B}+\boldsymbol{u}_0\cdot\boldsymbol{B}-\boldsymbol{u}_0\times\boldsymbol{E})(\beta\sinh\varepsilon s-\varepsilon\sin\beta s)\\
&\quad+\frac{1}{8\pi}[\gamma_0(U+\boldsymbol{S})+\boldsymbol{u}_0\cdot\bar{\boldsymbol{T}}-\boldsymbol{u}_0\cdot\boldsymbol{S}](\cosh\varepsilon s-\cos\beta s).\end{aligned}$$

3.3. *Arbitrarily polarised directed plane wave*

We consider the plane wave (9) with a well defined propagation vector $\boldsymbol{k}$ and note that the complex direction of $\boldsymbol{F}$ is constant, even in the case of a circularly polarised wave, for which $\boldsymbol{E}$ and $\boldsymbol{B}$ rotate about $\hat{k}$. As a result, a single integral is sufficient to determine the Lorentz transformation (40). In fact, since $\boldsymbol{F}^2 = 0$, $L(\tau)$ reduces to

$$L(\tau) = 1 + \frac{q}{4m}(\hat{\xi} + \mathrm{i}\hat{\eta}) \int_0^\tau \mathrm{d}\tau_1 f(\bar{k}\cdot x_1). \tag{47}$$

In the case of circularly polarised waves, there is a class of inertial frames in which the kinetic energy $(\gamma - 1)m$ of the charge is constant. The integral is then easily performed and the resulting 4-velocity (38) determined. For elliptically polarised waves, however, γ changes in time and another approach is simpler. The Pauli algebra permits a simplification and minor extension of the treatment by Landau and Lifshitz (1975).

Consider a 4-potential A whose time average vanishes ($\langle A\rangle = 0$) and which satisfies the Lorentz condition (3) and depends on the 4-position x through the scalar $\bar{k}\cdot x = \omega t - \boldsymbol{k}\cdot\boldsymbol{x}$. From the Lorentz-force equation in form (25), one sees that $\mathrm{d}\pi/\mathrm{d}\tau$ is a Lorentz-scalar function times $k = \omega + \boldsymbol{k}$:

$$\mathrm{d}\pi/\mathrm{d}\tau = q\partial(\bar{A}\cdot u) = qk(\bar{A}'\cdot u) \tag{48}$$

where $\bar{A}'$ is the derivative of $\bar{A}$ with respect to its argument $\bar{k}\cdot x$. It is apparent that the part $\boldsymbol{\pi}_\perp$ of the conjugate momentum perpendicular to $\hat{k}$ is constant:

$$\boldsymbol{\pi}_\perp = \boldsymbol{p}_\perp + q\boldsymbol{A}_\perp = \text{constant} \tag{49}$$

as is $\bar{k}\cdot\pi$ and hence $\bar{k}\cdot p$ (since $\bar{k}\cdot k = \bar{k}\cdot A = 0$):

$$E - \hat{k}\cdot\boldsymbol{p} \equiv \kappa = \text{constant} > 0. \tag{50}$$

If we choose a frame in which the time average $\langle \boldsymbol{p}_\perp\rangle = 0$, then

$$\boldsymbol{p}_\perp = -q\boldsymbol{A}_\perp \tag{51}$$

and

$$\begin{aligned} E + \hat{k}\cdot\boldsymbol{p} &= [E^2 - (\hat{k}\cdot\boldsymbol{p})^2]/\kappa \\ &= (q^2 A_\perp^2 + m^2)/\kappa. \end{aligned} \tag{52}$$

Subtraction of (50) from (52) yields

$$\hat{k}\cdot\boldsymbol{p} = \frac{1}{2\kappa}(q^2A_\perp^2 + m^2 - \kappa^2) \tag{53}$$

which, together with (51), gives the solution. The constant κ is given by the initial conditions. For example, in a frame in which $\langle \boldsymbol{p}\rangle = 0$, $\kappa^2 = \langle q^2 A_\perp^2\rangle + m^2$ and one finds the simple result

$$\hat{k}\cdot\boldsymbol{p} = \frac{q^2}{2\kappa}(A_\perp^2 - \langle A_\perp^2\rangle) = E - \kappa \qquad \boldsymbol{p}_\perp = -q\boldsymbol{A}_\perp. \tag{54}$$

3.4. Coulomb field

The field of a fixed charge Q at the origin is

$$\boldsymbol{F} = \boldsymbol{E} = Q\hat{r}/r^2. \tag{55}$$

Although $\boldsymbol{F}$ is static, $\boldsymbol{F}(\tau)$ at the position of the charge q changes in both magnitude and direction. We can find the appropriate Lorentz transformation by solving the differential equation (39), which with (55) has the form

$$\frac{\mathrm{d}L}{\mathrm{d}\tau} = \frac{qQ}{2mr^2}\,\hat{r}L. \tag{56}$$

The problem is considerably reduced by changing variables to the azimuthal angle θ:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{l}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\theta} \tag{57}$$

where l is the magnitude of the angular momentum

$$\boldsymbol{l} = l\hat{l} = \boldsymbol{r} \times m\boldsymbol{u} = \hat{l}mr^2\,\mathrm{d}\theta/\mathrm{d}\tau. \tag{58}$$

Combining (56) and (57) we find

$$\mathrm{d}L/\mathrm{d}\theta = \tfrac{1}{2}\alpha\hat{r}L \tag{59}$$

where $\alpha = qQ/l$. Now $\hat{r}$ is a function of θ:

$$\hat{r}(\theta) = \exp(-\mathrm{i}\hat{l}\theta/2)\hat{r}(0)\exp(\mathrm{i}\hat{l}\theta/2). \tag{60}$$

By extracting the rotation $\exp(-\mathrm{i}\hat{l}\theta/2)$ from $L(\theta)$ and putting

$$L(\theta) = \exp(-\mathrm{i}\hat{l}\theta/2)L_{\mathrm{r}}(\theta) \tag{61}$$

one easily obtains the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\theta}L_{\mathrm{r}}(\theta) = \tfrac{1}{2}\mathrm{i}\boldsymbol{\beta}L_{\mathrm{r}}(\theta) \tag{62}$$

where $\boldsymbol{\beta} = \hat{l} - \mathrm{i}\alpha\hat{r}(0)$, with the solution

$$L_{\mathrm{r}}(\theta) = \exp(\mathrm{i}\boldsymbol{\beta}\theta/2). \tag{63}$$

Instead of calculating $u(\theta)$ as in (38), we calculate u as seen in the frame rotated by θ about $\hat{l}$:

$$\exp(\mathrm{i}\hat{l}\theta/2)u(\theta)\exp(-\mathrm{i}\hat{l}\theta/2) = L_{\mathrm{r}}(\theta)u(0)L_{\mathrm{r}}^{+}(\theta) \tag{64}$$

which gives directly the radial (u_{r}) and tangential (u_{t}) components of u.

The calculation is straightforward. We decompose

$$u = \gamma + u_{\mathrm{r}}\hat{r} + u_{\mathrm{t}}\hat{l}\times\hat{r} \tag{65}$$

and expand

$$L_{\mathrm{r}}(\theta) = \cos(\beta\theta/2) + \mathrm{i}\boldsymbol{\beta}\frac{\sin(\beta\theta/2)}{\beta} \tag{66}$$

where $\beta = (\boldsymbol{\beta}^2)^{1/2} = (1-\alpha^2)^{1/2}$. The result is

$$\gamma(\theta) = \frac{A - \alpha\beta \cos \beta\theta}{\beta^2} + \alpha u_r(0) \frac{\sin \beta\theta}{\beta} \tag{67a}$$

$$u_r(\theta) = u_r(0) \cos \beta\theta + B \frac{\sin \beta\theta}{\beta} \tag{67b}$$

$$u_t(\theta) = \frac{B \cos \beta\theta - \alpha A}{\beta^2} - u_r(0) \frac{\sin \beta\theta}{\beta} \tag{67c}$$

where $A = \gamma(0) + \alpha u_t(0)$ and $B = \alpha\gamma(0) + u_t(0)$. In fact, from (67) we see that $\gamma(\theta) + \alpha u_t(\theta)$ is a constant.

The trajectories of the charge are easily found from the relation (see (58))

$$u_t(\theta) = \frac{l}{mr(\theta)}. \tag{68}$$

Combining this with (67c), one sees that the trajectories are conic sections in the angle variable $\beta\theta$ if $\alpha^2 < 1$. The relations (67) can then be further simplified by defining $\theta = 0$ to coincide with an apogee or perigee: $u_r(0) = 0$. The initial parameters A and B can then be expressed in terms of the total energy

$$E = \gamma m + \alpha l / r \tag{69}$$

by noting

$$u_t(0) = (\gamma^2(0) - 1)^{1/2} = l / mr(0). \tag{70}$$

One finds $A = E/m$ and $B = (A^2 - \beta^2)^{1/2}$, in agreement, for example, with Schwartz (1968).

The orbital precession predicted by special relativity is found from solutions to $u_r(\theta) = 0$ when $u_r(0) = 0$. One obtains additional perigees and apogees at

$$\beta\theta = (1-\alpha^2)^{1/2}\theta = n\pi. \tag{71}$$

The resulting precession for $\alpha^2 \ll 1$ is just one-sixth that predicted by general relativity (Einstein 1915, 1956) with $l\alpha = GMm$. Our solutions (67) should also be valid for $\alpha^2 > 1$ where β is pure imaginary. Then, however, there is only a single solution, namely $\theta = 0$, to $u_r(\theta) = 0$. For $\alpha \leqslant -1$, the trajectory is an inward spiral with ever-increasing kinetic energy: the charge q is captured if $l \leqslant -qQ$. For $-qQ = e^2$, this limit is well within the quantum regime: $l < \hbar/137$.

The Lorentz transformation $L_r(\theta)$ is neither a pure boost nor a pure rotation. However, it can always be written as a product of a pure boost and a pure rotation:

$$L_r(\theta) = \exp(\mathrm{i}\boldsymbol{\beta}\theta/2) = \exp(\mathrm{i}\boldsymbol{\Omega}/2) \exp(\boldsymbol{w}/2). \tag{72}$$

By expanding the exponentials as in equation (10) of Baylis and Jones (1989) and equating scalar, pseudoscalar, vector and pseudovector parts, we easily find

$$\hat{w} \sinh\tfrac{1}{2}w = \hat{r}\left(\frac{\Omega}{2}\right) \frac{\alpha}{\beta} \sin\tfrac{1}{2}\beta\theta \tag{73a}$$

$$\hat{\Omega} \tanh\tfrac{1}{2}\Omega = \frac{\hat{l}}{\beta} \tan\tfrac{1}{2}\beta\theta \tag{73b}$$

where $\hat{r}(\Omega/2)$ is the radial unit vector when $\theta = \Omega/2$ (see (60)). For small angles $\theta \ll 1$, the transformation parameters $\hat{w}$ and $\boldsymbol{\Omega}$ are small, and from (73b), $\boldsymbol{\Omega} \simeq \theta\hat{l}$. Consequently the full transformation (61) is nearly a pure boost.

At larger θ, $\boldsymbol{\Omega}$ and $\theta\hat{l}$ become distinct and L is a boost followed by a rotation $\theta - \Omega$ about $\hat{l}$. From (73b), the rotation, which includes effects of Thomas precession (Thomas 1926, 1927), is seen to be directly related to the orbital precession (see (71)). According to (73b), Ω must pass through a multiple of 2π every time $\beta\theta$ does. Because bound orbits, (67) and (68), repeat as $\beta\theta$ increases by 2π, the rotation angle $\theta - \Omega$ advances by $\theta - 2\pi$ every time θ increases by $2\pi/\beta$. Thus, for example, if we imagine the charge q to have internal structure, the same part will point toward the attracting centre at every perigee. Interpretation of the finite results is less subject to ambiguities than are the usual infinitesimal treatments of Thomas precession, where problems relating the infinitesimal quantities to cumulative rotations and boosts may arise from lack of commutivity.

The rotation predicted from (72) and (73) has the wrong sign and magnitude to be Thomas precession. In fact, it is the precession of a point charge and magnetic dipole with a g factor of $g = 2$, including the effects of Thomas precession. This surprising result will be discussed in more detail elsewhere.

4. Conclusions

One sees that the Pauli algebra shares much of the power and other advantages of larger Clifford algebras. Like the Dirac algebra, the Pauli algebra permits component-free covariant expressions and vector multiplication is associative and invertible. However, the Pauli algebra $\mathscr{P}$ is distinguished in being much simpler than its higher-order cousins. Indeed, it is the simplest Clifford algebra which includes both 3-space vectors and complex numbers. It requires only a minor extension of the mathematics familiar to all physicists (namely the algebraic multiplication of vectors, (3)) and the only outer (Grassmann) product needed is the usual cross product of 3-space vectors. All elements are complex linear combinations of scalars and vectors, and both real and imaginary parts of the scalars commute with all other elements.

When one wants to express elements in terms of components, there is a matrix representation of $\mathscr{P}$ in which every element is written as a 2×2 matrix. Like the algebra itself, it is efficient and free of the excess baggage of many zero or symmetric elements. In most applications, however, the algebra is so simple that the explicit use of components and matrices is unnecessary.

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